

# Fast Evaluation, Weights and Nonlinearity of Rotation-Symmetric Functions

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## Abstract

We study the nonlinearity and the weight of the *rotation-symmetric* (*RotS*) functions defined by Pieprzyk and Qu [6]. We give exact results for the nonlinearity and weight of 2-degree *RotS* functions with the help of the semi-bent functions [2] and we give the generating function for the weight of the 3-degree *RotS* function. Based on the numerical examples and our observations we state a conjecture on the nonlinearity and weight of the 3-degree *RotS* function.

*Keywords:* Boolean functions; nonlinearity; bent; semi-bent; hash functions

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## 1 Motivation

Hash functions are used to map a large collection of *messages* into a small set of *message digests* and can be used to generate efficiently both signatures and message authentication codes, and they can be also used as one-way

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functions in key agreement and key establishment protocols. There are two approaches to the study of hash functions: *Information Theory* and *Complexity Theory*. The first method provides unconditional security – an enemy cannot attack such systems even if he/she has unlimited computing power. Unfortunately, this is still a theoretical approach and is generally impractical [1]. In the second method based on complexity theory, some assumptions are made on the computing power of the enemy or the weaknesses of the existing systems and algorithms. The best we can hope for is to estimate the computing power necessary for the attacker to break the algorithm. Recent progress in interpolation cryptanalysis [4] and high order differential cryptanalysis [5] has shown that the algebraic degree is an important factor in the design of cryptographic primitives. In fact, in [5] the algebraic degree is the crucial parameter in determining how secure certain cryptosystems are against higher order differential attacks. Together with propagation, differential and nonlinearity profile, resiliency, correlation-immunity, local and global avalanche characteristics they form a class of design criteria which we have to consider in the design of such primitives.

In [6], Pieprzyk and Qu studied some functions, which they called *rotation-symmetric (RotS)* as components in the rounds of a hashing algorithm. It turns out that the degree-two *RotS* function takes  $\frac{3n-1}{2} + 6(m-1)$  opera-

tions (additions and multiplications) to evaluate in  $m$  consecutive rounds of a hashing algorithm. In [5] the authors showed how to break in less than 20 milli-seconds a block cipher that employs low algebraic degree (quadratic) Boolean functions as its S-boxes and is provably secure against linear and differential attacks. Therefore, it is necessary to employ high degree  $RotS$  functions in our algorithms. To protect from differential attack, we need  $RotS$  functions with high nonlinearity. In this paper we aim to complete the study begun by Pieprzyk and Qu [6] on the two-degree  $RotS$  functions and we construct the three-degree  $RotS$  functions and we prove some results about their weights and nonlinearity.

## 2 Preliminaries

Let  $n \geq 6$  be a positive integer and  $W_n = \{0, 1\}^n$  be the space of binary vectors. Denote  $\alpha_0 = (0, \dots, 0, 0)$ ,  $\alpha_1 = (0, \dots, 0, 1), \dots, \alpha_{2^n-1} = (1, \dots, 1)$ . We use the lexicographical order on the sequence  $\alpha$ , that is  $\alpha_0 < \alpha_1 < \dots < \alpha_{2^n-1}$ . The Boolean functions will be written in their algebraic normal form (when  $\alpha = (a_1, \dots, a_n)$ ) as

$$f(x) = \bigoplus_{\alpha \in W_n} c_\alpha x_1^{a_1} \cdots x_n^{a_n},$$

where  $c_\alpha \in W_1$ . The truth table of  $f$  is the binary sequence

$$f = (v_1, v_2, \dots, v_{2^n}), \quad (1)$$

where the bits  $v_1 = f((0, \dots, 0))$ ,  $v_2 = f((0, \dots, 0, 1))$ ,  $\dots$ . We shall identify the function  $f$  with its vector representation in (1). We call a function *balanced* if the number of ones is equal to the number of zeroes in its truth table. The *Hamming weight* of a binary vector  $v$ , denoted by  $wt(v)$  is defined as the number of ones it contains. The *Hamming distance* between two functions  $f, g : W_n \rightarrow W_1$ , denoted by  $d(f, g)$  is defined as  $wt(f \oplus g)$ . The nonlinearity of a function  $f$ , denoted by  $N_f$  is defined as

$$\min_{\phi \in A_n} d(f, \phi),$$

where  $A_n$  is the class of all affine function on  $W_n$ . We say that  $f$  satisfies the *propagation criterion (PC)* with respect to  $c$  if

$$\sum_{x \in \mathbf{Z}_2^n} f(x) \oplus f(x \oplus c) = 2^{n-1}. \quad (2)$$

If  $f$  satisfies the PC with respect to all vectors of weight 1,  $f$  is called an *SAC (Strict Avalanche Criterion)* function. If the above relation happens for any  $c$  with  $wt(c) \leq s$ , we say that  $f$  satisfies  $PC(s)$ , and if  $s = n$ , then we say that  $f$  is a *bent function*. If two functions  $g, h$ , on  $W_n$ , satisfy  $g(x) = h(Ax \oplus a) \oplus (b \cdot x) \oplus c$  with  $a, b \in W_n, c \in W_1$ , and  $A$  a  $2k \times 2k$  nonsingular matrix, we say that  $g$  is *affinely equivalent* to  $h$ .

**Definition 1.** *The class of rotation-symmetric (RotS) functions includes all Boolean functions  $f : W_n \rightarrow W_1$  such that  $f(x_1, \dots, x_n) = f(\rho(x_1), \dots,$*

$\rho(x_n)$ ), where  $\rho(x_i) = x_{i+1}$ , and  $x_{n+1} := x_1$ .

As in [6], we denote by  $\rho$  the permutation  $\rho(i) = i + 1, \rho(n) = 1$ . By abuse of notation we use the same letter for the transformation which acts on each variable by  $\rho(x_i) = x_{i+1}, \rho(x_n) = x_1$ . By  $\hat{g}$  we mean  $(-1)^g$ . We define the *Walsh-Hadamard transform* of a  $g \in W_n$  to be the map  $\hat{\mathcal{F}}_{\hat{g}} : W_n \rightarrow \mathbf{R}$ ,

$$\hat{\mathcal{F}}_{\hat{g}}(w) = \sum_{x \in W_n} \hat{g}(x) (-1)^{w \cdot x}.$$

The *correlation value* between  $g$  and  $h$  it is defined by

$$c(g, h) = 1 - \frac{d(g, h)}{2^{n-1}}.$$

If  $U$  is a string of bits, then  $\bar{U}$  denotes the complemented string with 0 and 1 interchanged. If  $X$  is a 4-bit block or a string of blocks, by  $(X)_u$  or  $X_u$  we shall mean the string obtained by concatenation of  $u$  copies of  $X$ . The concatenation of two strings  $u, v$  will be denoted by  $uv$  or  $u||v$ . Now we define two sets of 4-bit strings

$$T_1 = \{A = 0, 0, 1, 1; \bar{A} = 1, 1, 0, 0; B = 0, 1, 0, 1; \bar{B} = 1, 0, 1, 0; \\ C = 0, 1, 1, 0; \bar{C} = 1, 0, 0, 1; D = 0, 0, 0, 0; \bar{D} = 1, 1, 1, 1\}$$

and

$$T_2 = \{U = 1, 0, 0, 0; \bar{U} = 0, 1, 1, 1; V = 0, 0, 0, 1; \bar{V} = 1, 1, 1, 0; \\ X = 0, 1, 0, 0; \bar{X} = 1, 0, 1, 1; Y = 0, 0, 1, 0; \bar{Y} = 1, 1, 0, 1\}.$$

### 3 The second degree rotation-symmetric function

In [6] the authors proved that the homogeneous rotation symmetric function of degree 2,  $f_2 = x_1x_l + x_2x_{l+1} + \cdots + x_nx_{n+l-1}$ , (the subscript  $w$  is taken as  $((w-1) \bmod n) + 1$ ) has good nonlinearity and good avalanche properties. Precisely, they proved

**Theorem 2.** *The function  $f_2$  has the following properties:*

- (i) *the Hamming weight satisfies  $2^{n-2} \leq \text{wt}(f) \leq 2^n - 2^{n-2}$ ,*
- (ii) *the nonlinearity satisfies  $N_{f_2} \geq 2^{n-2}$ ,*
- (iii) *if  $n$  is odd, then  $N_{f_2} = 2^{n-1} - 2^{\frac{n-1}{2}}$  and  $f_2$  is balanced,*
- (iv) *the function satisfies the PC with respect to all vectors  $\alpha$  of weight  $0 < \text{wt}(\alpha) < n$ . In particular  $f$  is an SAC function.*

In the same paper it is proved that

**Theorem 3.** *If  $f_k$  is an RotS function of degree  $k$ , then the nonlinearity satisfies  $N_{f_k} \geq 2^{n-k}$ .*

Now, we evaluate the nonlinearity of  $f_2$  for  $n$  even.

**Lemma 4.** *For  $n \geq 3$ , let  $t_n = x_1x_2 + x_2x_3 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_n$ . Then  $t_{2k}$  is a bent function.*

**Proof.** We have

$$\begin{aligned} t_{2k} = & x_2(x_1 + x_3) + x_4(x_3 + x_5) + \cdots + \\ & x_{2k-2}(x_{2k-3} + x_{2k-1}) + x_{2k}x_{2k-1}. \end{aligned}$$

By taking the transformation

$$X_{2i} = x_{2i} \text{ and } X_{2i-1} = x_{2i-1} + x_{2i+1}, X_{2k-1} = x_{2k-1}, i = 1, 2, \dots, k-1,$$

we see that  $t_{2k}$  is affinely equivalent to a bent function in the Maiorana-McFarland class (see [3]), therefore it is also bent.  $\square$

We say (see [2]) that  $g \in W_{2k+1}$  is *semi-bent*, if there is a bent function  $g_0 \in W_{2k}$  with

$$g = g_0 || g_1,$$

where  $g_1(x) = g_0(Ax \oplus a) \oplus 1$ ,  $A$  is a nonsingular  $2k$  by  $2k$  matrix and  $a$  is any vector in  $W_{2k}$ .

In [2], the authors prove the following results (see Theorem 18, Corollary 21 and Theorem 16), which will be used in this paper.

**Lemma 5.** *Any semi-bent function  $g \in W_{2k+1}$  is balanced,  $N_g = 2^{2k} - 2^k$ ,*

*for any  $w^* \in W_{2k+1}$ , the correlation value between  $g$  and the linear function*

*$l_{w^*}(x) = w^* \cdot x$  is 0 or  $\pm 2^{-k}$ , and*

$$\#\{w^* \in W_{2k+1} \mid c(g, l_{w^*}) = 0\} = 2^{2k} = \#\{w^* \in W_{2k+1} \mid c(g, l_{w^*}) = \pm 2^{-k}\}.$$

**Lemma 6.** *Let  $g \in W_{2k+1}$  be a semi-bent function with  $A = I$  and  $a = (1, 1, \dots, 1)$ . Then  $g$  satisfies  $PC(2k)$ .*

**Lemma 7.** *If  $g$  is the concatenation  $g_0||g_1$ ,  $w^* = (w, w_{2n+1}) \in W_{n+1}$ , then*

$$\hat{\mathcal{F}}_{\hat{g}}(w^*) = \hat{\mathcal{F}}_{\hat{g}_0}(w) + (-1)^{w_{n+1}} \hat{\mathcal{F}}_{\hat{g}_1}(w).$$

The following result belongs to Preneel [7]. We define  $l_b(x) = b \cdot x$ .

**Lemma 8.** *For  $h$  on  $W_n$ ,  $a, b \in W_n$ ,  $c \in W_1$  and a  $2k \times 2k$  nonsingular matrix  $A$ , define  $g$  by  $g(x) = h(Ax \oplus a) \oplus l_b(x) \oplus c$ . Then,*

$$\hat{\mathcal{F}}_{\hat{g}}(w) = (-1)^c (-1)^{(A^{-1}a, w \oplus b)} \hat{\mathcal{F}}_h((A^{-1})^t(w \oplus b)).$$

It is not very difficult to observe (see also [6]) that any 2-degree rotation-symmetric function in  $n$  variables is affinely equivalent to  $f_2^n = f_2 = x_1x_2 \oplus x_2x_3 \oplus \dots \oplus x_{n-1}x_n \oplus x_nx_1$ . We show below that  $f_2^{2k}$  is not bent. To do that we display an algorithm to evaluate  $f_2^{2k}$  fast. For that we need the following lemma, which can be proved by considering the truth table.

**Lemma 9.** *Each monomial of degree 2 can be written in the form (1) as*

$$\begin{aligned}
 x_i x_j &= (D_{2^{n-i-2}} (D_{2^{n-j-2}} \bar{D}_{2^{n-j-2}})_{2^{j-i-1}})_{2^{i-1}}, \\
 &\quad \text{if } 1 \leq i < j \leq n-2, \\
 x_i x_{n-1} &= (D_{2^{n-i-2}} A_{2^{n-i-2}})_{2^{i-1}}, \\
 x_i x_n &= (D_{2^{n-i-2}} B_{2^{n-i-2}})_{2^{i-1}}, \\
 x_{n-1} x_n &= V_{2^{n-2}}.
 \end{aligned} \tag{3}$$

Using (3) we see that

$$\begin{aligned}
 f_2 &= x_1 x_2 \oplus x_2 x_3 \oplus \cdots x_{n-1} x_n \oplus x_n x_1 = \\
 &= D_{2^{n-3}} (D_{2^{n-4}} \bar{D}_{2^{n-4}}) \oplus (D_{2^{n-4}} (D_{2^{n-5}} \bar{D}_{2^{n-5}}))_2 \oplus \dots \tag{4} \\
 &\quad (D_2 (D \bar{D}))_{2^{n-4}} \oplus (DA)_{2^{n-3}} \oplus (DB)_{2^{n-3}} \oplus V_{2^{n-2}} = \\
 &= g \oplus (DC)_{2^{n-3}} \oplus V_{2^{n-2}} = g \oplus (V \bar{U})_{2^{n-3}} \oplus D_{2^{n-3}} B_{2^{n-3}},
 \end{aligned}$$

where  $g$  is the sum of the first  $n-3$  strings of length  $2^n$ .

For a string  $u$  of length  $2^s$ ,  $s \geq 4$ , we denote by  $\tilde{u}$ , the string obtained by complementing the second half, that is the last  $2^{s-1}$  bits of  $u$ . It is not difficult to observe that the following algorithm will output  $f_2 = G_1 || G_2 || G_3$ .

**Algorithm f2.**

**step 3:**  $g_1^3 \leftarrow VY, g_2^3 \leftarrow X\bar{U}$

**step  $s$ :**  $g_i^s \leftarrow g_i^{s-1} || \tilde{g}_i^{s-1}, i = 1, 2$

**output:**  $G_1 \leftarrow g_1^{n-4}, G_2 \leftarrow g_2^{n-5}, G_3 = \bar{G}_4$ , where  $G_4 = \tilde{G}_2$ , and write

$$f_2 = G_1 \parallel G_2 \parallel G_3$$

For instance, the first three steps of the algorithm will produce

$$\begin{aligned} G_1 &\leftarrow ((VY)(V\bar{Y}))((VY)(\bar{V}Y)) \\ G_2 \parallel G_3 &\leftarrow (X\bar{U}XU)(\bar{X}UXU). \end{aligned}$$

**Theorem 10.** *If  $f_2$  is defined on  $W_n$ , with  $n = 2k$ , then it is not bent.*

Moreover, the nonlinearity is

$$N_{f_2} = 2^{2k-1} - 2^k,$$

and the truth table of  $f_2$  can be displayed using only  $2^{n-3} - 2$  operations (additions and multiplications).

**Proof.** Using the above algorithm, we deduce that the *RotS* function on  $W_n$  of degree 2 can be evaluated in  $n - 2$  steps, which requires

$$(1 + 2^1 + \cdots + 2^{n-5}) + (1 + 2^1 + \cdots + 2^{n-6}) + 2^{n-5} = 2^{n-3} - 2$$

operations, since at each step  $s$  we complement  $2^{s-2}$  bits.

First, we take an example, say  $f_2^5 = VYV\bar{Y}X\bar{U}\bar{X}\bar{U} = t_5 + x_1x_5$  on  $W_5$ .

We see that  $f_2^5 = t_4(\mathbf{x}) \parallel t_4(\mathbf{x} \oplus \mathbf{1}) \oplus 1$ , therefore it is semi-bent.

It is very easy to see that

$$f_2^{2k+1} = t_{2k}(\mathbf{x}_{2k}) \parallel (t_{2k}(\mathbf{x}_{2k}) + x_1 + x_{2k}).$$

But

$$\begin{aligned} \overline{t_{2k}(\mathbf{x}_{2k}) + x_1 + x_{2k}} &= \sum_{i=1}^{2k-1} x_i x_{i+1} + x_1 + x_{2k} + 1 = \\ &= \sum_{i=1}^{2k-1} (x_i + 1)(x_{i+1} + 1) = t_{2k}(\mathbf{x}_{2k} \oplus \mathbf{1}), \end{aligned}$$

therefore  $f_2^{2k+1} = t_{2k}(\mathbf{x}_{2k})||(t_{2k}(\mathbf{x}_{2k} \oplus \mathbf{1}) \oplus 1)$  is semi-bent. By Lemma 6,

$f_2^{2k+1}$  satisfies the propagation criterion for all weights  $1 \leq w \leq 2k$ .

Similarly,

$$f_2^{2k} = t_{2k-1}(\mathbf{x}_{2k-1})||(t_{2k-1}(\mathbf{x}_{2k-1}) + x_1 + x_{2k-1}).$$

Now, we shall use Lemma 7 to compute the nonlinearity of  $f_2^{2k}$ . First,

we observe that

$$t_{2k+1} = t_{2k}(\mathbf{x}_{2k})||(t_{2k}(\mathbf{x}_{2k}) + x_{2k}).$$

Take  $A = I$  and  $a = (1, 0, 1, 0, \dots, 1, 0)$ . We see that

$$\begin{aligned} t_{2k}(\mathbf{x}_{2k}) + x_{2k} &= t_{2k}(x_1 + 1, x_2, x_3 + 1, \dots, x_{2k-1}, x_{2k}) \\ &= (x_1 + 1)x_2 + x_2(x_3 + 1) + \dots + (x_{2k-1} + 1)x_{2k}. \end{aligned}$$

We denote the last expression by  $r(\mathbf{x})$ . Using Lemma 7 we compute the Walsh-Hadamard transform

$$\hat{\mathcal{F}}_{\hat{r}}(\mathbf{w}_{2k}) = (-1)^{(\mathbf{w}, \mathbf{a})} \hat{\mathcal{F}}_{t_{2k}}(\mathbf{w}) = \pm 2^k,$$

since by Lemma 4,  $t_{2k}$  is bent.

For simplicity we set  $t(\mathbf{x}) = t_{2k+1}(\mathbf{x}_{2k+1})$  and  $w^* = (w, w_{2k+1})$ . Thus,

$$\hat{\mathcal{F}}_t(w^*) = \hat{\mathcal{F}}_{\hat{t}_{2n}}(w) + (-1)^{w_{2k+1}} \hat{\mathcal{F}}_r(w) = 0 \text{ or } \pm 2^{k+1}, \quad (5)$$

since  $r$  and  $t_{2k}$  are bent. Therefore,

$$N_{t_{2k+1}} = 2^{2k} - \frac{1}{2} |\hat{\mathcal{F}}_{\hat{t}_{2k+1}}(w^*)| = 2^{2k} - 2^k.$$

By Lemma 5,  $\hat{\mathcal{F}}_{\hat{t}_{2k-1}}(\mathbf{x}_{2k-1}) = 0$  or  $\pm 2^k$ . Let  $v(\mathbf{x}) = t_{2k-1}(\mathbf{x}) + x_1 + x_{2k-1}$ .

By Lemma 7,

$$\hat{\mathcal{F}}_{\hat{v}}(\mathbf{x}) = (-1)^{(\mathbf{x} \oplus (1, 0, \dots, 0, 1), \mathbf{0})} \hat{\mathcal{F}}_{\hat{t}_{2k-1}}(\mathbf{x} \oplus (1, 0, \dots, 0, 1)) = 0 \text{ or } \pm 2^k.$$

Thus, by the same Lemma 7,

$$\hat{\mathcal{F}}_{\hat{f}_2^{2k}}(\mathbf{x}_{2k}) = \hat{\mathcal{F}}_{\hat{t}_{2k-1}}(\mathbf{x}_{2k-1}) + (-1)^{x_{2k}} \hat{\mathcal{F}}_{\hat{v}}(\mathbf{x}) = 0 \text{ or } \pm 2^{k+1},$$

which implies  $N_{f_2^{2k}} = 2^{2k-1} - 2^k$ . Therefore  $f_2$  is not bent (any bent function in  $2k$  variables has nonlinearity  $2^{2k-1} - 2^{k-1}$  [2, Th. 13, p. 111]) and the theorem is proved.  $\square$

**Remark 11.** *We remark that, using the normal form of the function, the truth table of  $f$  is found using  $\frac{3n-1}{2}2^n$  operations (see [6] for a detailed discussion). Using the previous theorem we can display the truth table using only  $2^{n-3} - 2$  operations, which is a significant improvement.*

Now, we will evaluate the weights of  $f_2$  for any dimension  $n$ . We prove

**Theorem 12.** *The weights of  $f_2$  are given by*

$$wt(f_2^n) = 2^{n-1} - 2^{\frac{n}{2}-1} (1 + (-1)^n). \quad (6)$$

**Proof.** We recall that  $f_2 = g_1^{n-1} g_2^{n-2} g_3^{n-2}$ . We show that for any  $s$ ,

$$wt(g_i^s) = 2wt(g_i^{s-2}) + 2^{s-2}, i = 1, 2, 3. \quad (7)$$

$$\text{Since } g_i^s = g_i^{s-1} \tilde{g}_i^{s-1} = g_i^{s-1} g_i^{s-2} \tilde{g}_i^{s-2},$$

$$\begin{aligned} wt(g_i^s) &= wt(g_i^{s-1}) + wt(g_i^{s-2}) + wt(\tilde{g}_i^{s-2}) \\ &= wt(g_i^{s-2}) + wt(\tilde{g}_i^{s-2}) + wt(g_i^{s-1}) + 2^{s-2} - wt(\tilde{g}_i^{s-2}) \quad (8) \\ &= 2wt(g_i^{s-2}) + 2^{s-2}, i = 1, 2. \end{aligned}$$

Now, from  $g_3^s = \bar{g}_2^{s-1} \tilde{g}_2^{s-1}$ , we get

$$\begin{aligned} wt(g_3^s) &= 2^{s-1} - wt(g_2^{s-1}) + wt(\tilde{g}_2^{s-1}) \\ &= 2^{s-1} - wt(g_2^{s-1}) + 2wt(g_2^{s-2}) - wt(g_2^{s-1}) + 2^{s-2} \quad (9) \\ &= 2wt(g_2^{s-2}) - 2wt(g_2^{s-1}) + 2^{s-1} + 2^{s-2} \\ &= wt(g_2^s) - 2wt(g_2^{s-1}) + 2^{s-1}. \end{aligned}$$

The above equation, for  $s - 1$ , produces

$$wt(g_3^{s-1}) = wt(g_2^{s-1}) - 2wt(g_2^{s-2}) + 2^{s-2}. \quad (10)$$

Now, we add (9) plus twice (10), and we get

$$wt(g_3^s) + 2wt(g_3^{s-1}) = wt(g_2^s) - 4wt(g_2^{s-2}) + 2^s.$$

But  $wt(g_2^s) = 2wt(g_2^{s-2}) + 2^{s-2}$ . By adding the two previous equations we get

$$wt(g_2^{s-2}) = 2^{s-1} + 2^{s-3} - wt(g_3^{s-1}) - \frac{wt(g_3^s)}{2} \quad (11)$$

Replacing (11) into (10) we obtain

$$wt(g_3^{s+2}) = 2wt(g_3^s) + 2^s.$$

This together with (8) will give the following recurrence for the weights of  $f_2$ .

$$wt(f_2^n) = 2wt(f_2^{n-2}) + 2^{n-2}. \quad (12)$$

A generating function for the above recurrence is

$$-\frac{32 \frac{z^7}{1-2z} + 16z^5 + 24z^6}{-1+2z^2}. \quad (13)$$

We can linearize the recurrence by using the transformation

$$y_n = wt(f_2^n) - 2^{n-1},$$

thus obtaining the recurrence

$$y_n = 2y_{n-2}.$$

Using the above simple recurrence with  $wt(f_2^5) = 16$  and  $wt(f_2^6) = 24$  we get a closed formula for the weights of  $f_2$  in dimension  $n$ , namely

$$2^{n-1} - 2^{\frac{n}{2}-1} (1 + (-1)^n),$$

and the theorem is proved.  $\square$

## 4 The third degree rotation-symmetric function

As in the case of second degree *RotS* functions, it is easy to observe that any *RotS* function of degree 3 in  $n$  variables,  $f_3^n = f_3$  is affinely equivalent to

$$f_3 = x_1x_2x_3 + x_2x_3x_4 + \cdots + x_nx_1x_2. \quad (14)$$

Now, using a computer program we have determined the nonlinearity of  $f_3$  on  $W_n, n \geq 9$  which turns out to be the same as its weight. Thus,

$n$	3	4	5	6	7	8	9
$N_{f_3^n}$	1	4	6	18	36	80	172

We shall assume that  $n \geq 10$ . The following lemma will be used.

**Lemma 13.** *The truth table of any monomial  $x_{i_1} \cdots x_{i_s}$  of degree  $s$  is*

$$\begin{aligned} & \left( D_{2^{n-i_1-2}} \cdots \left( D_{2^{n-i_s-2}} \bar{D}_{2^{n-i_s-2}} \right)_{2^{i_s-i_{s-1}-1}} \right)_{2^{i_1-1}}, \\ & \text{if } 1 \leq i_1 < \cdots < i_s \leq n-2, \\ & \left( D_{2^{n-i_1-2}} \cdots \left( D_{2^{n-i_{s-1}-2}} M_{2^{n-i_{s-1}-2}} \right)_{2^{i_{s-1}-i_{s-2}-1}} \right)_{2^{i_1-1}}, \end{aligned} \quad (15)$$

where  $M = A$  or  $B$  if  $i_s = n-1$ , respectively  $i_s = n$ ,

$$\begin{aligned} & \left( D_{2^{n-i_1-2}} \cdots \left( D_{2^{n-i_{s-2}-2}} V_{2^{n-i_{s-2}-2}} \right)_{2^{i_{s-2}-i_{s-3}-1}} \right)_{2^{i_1-1}}, \\ & \text{if } i_{s-1} = n-1 \text{ and } i_s = n. \end{aligned}$$

**Proof.** Straightforward using the truth table.  $\square$

Using the above lemma we write

$$x_i x_{i+1} x_{i+2} = (D_{2^{n-i-2}} (D_{2^{n-i-3}} (D_{2^{n-i-4}} \bar{D}_{2^{n-i-3}})))_{2^{i-1}}, \quad (16)$$

if  $i \leq n-4$ , and

$$\begin{aligned} x_{n-3} x_{n-2} x_{n-1} &= (D_3 A)_{2^{n-4}} \\ x_{n-2} x_{n-1} x_n &= (D V)_{2^{n-3}} \\ x_{n-1} x_n x_1 &= D_{2^{n-3}} V_{2^{n-3}} \\ x_n x_2 x_1 &= D_{2^{n-3}+2^{n-4}} B_{2^{n-4}} \end{aligned} \quad (17)$$

Therefore,

$$\begin{aligned} f_3 &= \sum_{i=1}^{n-4} (D_{2^{n-i-2}+2^{n-i-3}+2^{n-i-4}} \bar{D}_{2^{n-i-4}})_{2^{i-1}} \oplus (D_3 A)_{2^{n-4}} \oplus \\ &\quad (D V)_{2^{n-3}} \oplus D_{2^{n-3}+2^{n-4}} B_{2^{n-4}} \oplus D_{2^{n-3}} V_{2^{n-3}} = \\ &= \sum_{i=1}^{n-4} (D_{2^{n-i-2}+2^{n-i-3}+2^{n-i-4}} \bar{D}_{2^{n-i-4}})_{2^{i-1}} \oplus \\ &\quad (D V D Y)_{2^{n-4}} \oplus D_{2^{n-3}} V_{2^{n-4}} X_{2^{n-4}} = \\ &= \sum_{i=1}^{n-4} (D_{2^{n-i-2}+2^{n-i-3}+2^{n-i-4}} \bar{D}_{2^{n-i-4}})_{2^{i-1}} \oplus \\ &\quad (D_3 C)_{2^{n-5}} (V_3 \bar{U})_{2^{n-6}} (X_3 Y)_{2^{n-6}} = H_1 || H_2 || H_3 || H_4, \end{aligned}$$

where  $H_1$  (on  $W_{n-1}$ ),  $H_2$  (on  $W_{n-2}$ ),  $H_3, H_4$  (on  $W_{n-3}$ ) are defined by the following algorithm ( $\hat{u}$ , on  $W_j$ , is the string obtained from  $u$  by complementing its last  $2^{j-2}$  bits):

**Algorithm f3.**

**step 4:**  $h_1^4 \leftarrow DV DY$   $h_2^4 \leftarrow DV VA$ ,  $h_3^4 \leftarrow XB XC$

**step s:**  $h_i^s \leftarrow h_i^{s-1} \parallel \hat{h}_i^{s-1}$

**output:**  $H_1 \leftarrow h_1^{n-1}, H_2 \leftarrow h_2^{n-2}, H_3 \leftarrow h_3^{n-3}$ ,  $H_4$  is the string obtained from  $\hat{H}_3$  by complementing its first half, that is  $H_4 = \bar{H}_5$ , where  $H_5 = \tilde{H}_6, H_6 = \hat{H}_3$ . Write  $f_3 = H_1 \parallel H_2 \parallel H_3 \parallel H_4$ .

As in the case of the 2-degree *RotS* function we see that we need

$$\begin{aligned} & 2^2(1 + 2 + \cdots 2^{n-5}) + 2^2(1 + 2 + \cdots 2^{n-6}) + \\ & 2^2(1 + 2 + \cdots 2^{n-7}) + 2^{n-4} + 2^{n-5} = 3 \cdot 2^2(2^{n-6} - 1) + \\ & 2^{n-3} + 2^{n-4} = 2^{n-2} + 2^{n-4} + 2^{n-5} - 3 \cdot 2^2 \end{aligned}$$

operations to display the truth table of  $f_3^n$ .

We shall evaluate the weight of  $f_3^s$  for any  $s$ . To do this we will compute the weights of each component of  $f_3^s$ . We observe that

$$\begin{aligned} h_i^s &= h_i^{s-1} h_i^{s-2} h_i^{s-3} \bar{h}_i^{s-4} \hat{h}_i^{s-4} \text{ and} \\ \hat{h}_i^s &= h_i^{s-1} h_i^{s-2} \bar{h}_i^{s-3} h_i^{s-4} \hat{h}_i^{s-4}, i = 1, 2, 3. \end{aligned}$$

Therefore, denoting by  $w_i^s$  the weight of  $h_i^s$ , and by  $\hat{w}_i^s$  the weight of  $\hat{h}_i^s, i = 1, 2, 3$ , we arrive at the following identities:

$$\hat{w}_i^s = 2w_i^{s-1} + 2w_i^{s-2} - w_s + 2^{s-2}, \quad (18)$$

$$w_i^s = w_i^{s-1} + \hat{w}_i^{s-1}. \quad (19)$$

Using Mathematica<sup>1</sup> we obtained the following results on the weights of  $f_3^n$  and of each of the four components on dimensions less than 12.

$n$	$wt(f_3^n)$	$wt(h_1^{n-1})$	$wt(h_2^{n-2})$	$wt(h_3^{n-3})$	$wt(h_4^{n-3})$	
3	1					
4	4					
5	6	2				
6	18	6	4			
7	36	14	8	6	8	
8	80	32	18	12	18	
9	172	72	40	26	34	
10	360	156	84	52	68	
11	760	336	180	108	136	
12	1576	712	376	220	268	

(20)

We have

$$wt(f_3^n) = wt(h_1^{n-1}) + wt(h_2^{n-2}) + wt(h_3^{n-3}) + wt(h_4^{n-3}).$$

We show by induction that

$$wt(h_i^s) = 2(wt(h_i^{s-2}) + wt(h_i^{s-3})) + 2^{s-4}, i = 1, 2, 3, 4. \quad (21)$$

From the table (20) we have the truth of the claim for the first few cases.

Assume (21) true for  $s - 1$  and we prove it for  $s$ . From (18) and (19) and by using the induction step we get

$$wt(h_i^s) = wt(h_i^{s-1}) + wt(\hat{h}_i^{s-1}) = 2(wt(h_i^{s-2}) + wt(h_i^{s-3})) + 2^{s-3}.$$

Similarly for  $h_4^s$ . Adding these relations we get

$$wt(f_3^s) = 2(wt(f_3^{s-2}) + wt(f_3^{s-3})) + 2^{s-3}. \quad (22)$$

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<sup>1</sup>A trademark of *Wolfram Research*

Remark that this equation is true for any  $s \geq 6$ .

Using the table (20), the recurrence (22) and Maple<sup>2</sup>, we get

**Theorem 14.** *The generating function for the weight of  $f_3$ , is*

$$-\frac{8 \frac{z^6}{1-2z} + z^3 + 4z^4 + 4z^5}{-1 + 2z^2 + 2z^3}. \quad (23)$$

The series expansion of the above generating function is

$$\begin{aligned} & z^3 + 4z^4 + 6z^5 + 18z^6 + 36z^7 + 80z^8 + \\ & + 172z^9 + 360z^{10} + 760z^{11} + 1576z^{12} + O(z^{13}), \end{aligned}$$

obtaining once again the weights of  $f_3^n$ , for any dimension.

Based on our numerical examples, we give the following conjecture.

**Conjecture 15.** *The nonlinearity of  $f_3^n$  is the same as its weight.*

## References

- [1] S. Bakhtiari, R. Sahavi-Naini, J. Pieprzyk, Cryptographic Hash Functions: A Survey, Preprint 95-9, Department of Computer Science, The University of Wollongong, 1995.
- [2] S. Chee, S. Lee, K. Kim, Semi-bent functions, Adv. in Cryptology - Asiacrypt' 94, LNCS 917, Springer-Verlag, 1995, pp. 107-118.

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<sup>2</sup>A trademark of Waterloo Maple

- [3] J.F. Dillon, A survey of bent functions, NSA Technical Journal—unclassified (1972), pp. 191-215.
- [4] T. Jakobsen, L. Knudsen, The interpolation attack on block ciphers, *Fast Software Encryption*, LNCS 1267 Springer Verlag, 1997, pp. 28-40.
- [5] S. Moriai, T. Shimoyama and T. Kaneko, Higher order differential attack using chosen higher order differences, *Selected Areas in Cryptography - SAC '98*, LNCS 1556, Springer Verlag, 1999, pp. 106-117.
- [6] J. Pieprzyk, C.X. Qu, Fast Hashing and Rotation-Symmetric Functions, *Journal of Universal Computer Science* **5**, no. 1 (1999), pp. 20-31.
- [7] B. Preneel, Analysis and design of cryptographic hash functions, Ph.D. dissertation, Katholieke Universiteit Leuven, 1993.
- [8] J. Seberry, X.-M. Zhang, Y. Zheng, Nonlinearity and Propagation Characteristics of Balanced Boolean Functions, *Information and Computation* **119**, no. 1 (1995), pp. 1-13.

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